A Theory of Laminar Flow Stability

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The success enjoyed by a laminar flow stability parameter that I previously developed as a generalization of the critical Reynolds number of laminar turbulent transition has occasioned considerable interest in the phenomenological theory underlying the parameter. In this paper an analysis of laminar flow stability is presented which leads naturally to the parameter in a much different manner than originally proposed. The stability parameter is seen to represent the coupling ratio between the rate of change of angular momentum of a deforming fluid element and its rate of loss of momentum by frictional drag. At a certain critical value of this coupling ratio, the element becomes unstable to rotational disturbances. If such disturbances are present, the basic nonlinearity of the momentum transfer process guarantees rapid amplification and generation of a turbulent eddy. The consequences of the theory are examined for two special fixed boundary classes of motion. The physical interpretation of the parameter is compared with conventional interpretations of the Reynolds number and found to be more fundamentally sound. The application of the theory to moving boundary flows, such as the Couette viscometer, is also discussed and an important physical difference is pointed out.

The origin of turbulent eddies is a very complex phenomenon, the complete description of which has thus far defied all efforts. However, much is known about various aspects of this highly nonlinear phenomenon.

Observations of boundary layers (*1, 2*) have revealed that the source of the turbs in a turbulent stream is in the boundary layer near the wall. Evidently, as this layer becomes unstable, certain disturbances grow and erupt violently into large scale eddies. This process is apparently linear at first but rapidly becomes nonlinear.* With the eruption of the turb the local fluid region experiences large acceleration forces and restabilizes, terminating the turbulent energy transport. The restabilized boundary layer again builds up in energy content until a condition of dynamic instability develops and the entire cycle is repeated. Consequently, the generation of turbulent eddies occurs in a series of bursts (*3*) which subsequently decay into fine scale eddies.

The phenomenon of transition apparently involves the interaction of three mechanisms: development of velocity profile instability, disturbance inception, and turbulent energy transport and eddy decay with boundary layer restabilization. The simultaneous occurrence of all three mechanisms is a necessary and sufficient condition for transition. If any of the three is absent, laminar flow prevails. For example, a low Reynolds number parabolic velocity field is stable and no amount of disturbances can be sustained as turbulent motion. Similarly, although a very high Reynolds number parabolic velocity field is unstable, by removing all sources of disturbance it is possible to maintain it in laminar motion. Finally, the removal of turbulent energy transfer mechanisms by means of acceleration of the flow (*4*) can laminarize an otherwise normally turbulent fluid.

Of the essence in all the above situations, however, is the stability of the velocity field. In the large majority of practical flow devices, ample disturbance levels and turbulent energy transfer situations exist so that the occurrence of transition is governed by the occurrence of the proper type of instability in the flow field. It is with this phenomenon and the factors governing its development that we are here concerned.

* The nonlinearity referred to here is the basic nonlinearity of the momentum conservation equations with respect to velocity components. Where unsteady state (with respect to time) analyses are made of disturbances, the equations of motion are usually linearized by one means or another to eliminate the nonlinear terms. However, when accelerations become large, the nonlinear terms become predominant and cannot be ignored. Conventional stability analyses become extremely difficult where the nonlinear effects are included.

THEORY

Many theories of laminar flow stability have been proposed. One class of theories summarized by Lin (*5*) consists of an analysis of the response of linearized forms of the differential equations of motion to certain time dependent velocity fluctuations. Such perturbation approaches have enjoyed varying degrees of success (*5*).

A second class of stability theories might be classed as phenomenological in that they involve the formulation of parametric ratios of physical terms related to stability (*6* to *8*). We shall present below an analysis which leads to one of these phenomenological theories. The theory was originally developed (*8*) simply as a generalization of the critical pipe flow Reynolds number for laminar-turbulent transition. However, it has since proven to be remarkably successful in predicting transition (*9* to *15*) thus suggesting a more fundamental basis than originally proposed.

If we consider a material element (*16*) of a fluid, we discover that its momentum, m^j, is given by

$$m^j = \int_V \rho v^j dV \tag{1}$$

where V is the material volume, ρ is the fluid density, and v^j is the velocity field vector. Newton's second law of motion states that the time rate of change of m^j is equal to the resultant force acting on the material element. This force is (*16*)

$$R^j = \int_V \rho f^j dV + \int_S T^{ji} n_i dS \tag{2}$$

where f^j is the external body force per unit mass and T^{ji} is the contravient stress field tensor giving rise to a tractive force $T^{ji}n_i$ on the element of surface dS having a covarient unit outward normal vector, n_i.

The rate of change of momentum, $\delta m^j/\delta t$, can be expressed by using Reynolds' transport theorem (*16*) as†

$$\frac{\delta m^j}{\delta t} = \int_V \left\{ \frac{\partial}{\partial t}(\rho v^j) + (\rho v^j v^k)_{,k} \right\} dV \tag{3}$$

where $\rho v^j v^k$ is the momentum flux tensor and represents the flux of momentum ρv^j through an area having a normal n_k.

The integrand function in Equation (3) can be expressed in a more useful form by use of the equation of continuity as

$$\frac{\partial}{\partial t}(\rho v^j) + (\rho v^j v^k)_{,k} = \rho \frac{\partial v^j}{\partial t} + \frac{1}{2} \rho g^{jk} (v^2)_{,k} + \rho g^{jk} \epsilon_{kpq} w^p v^q \tag{4}$$

† Here by $\delta m^j/\delta t$ we imply intrinsic differentiation[16], $\delta m^j/\delta t = \partial m^j/\partial t + v^k m^j_{,k}$.

where $v^2 = g_{ij}v^iv^j$, $w^p = \epsilon^{pqr} g_{rs}v^s{}_{,q}$, is the vorticity of the flow and g^{jk} and g_{jk} are the contravarient and covarient forms of the metric tensor. The vorticity vector, w^j, represents twice the angular velocity of rotation of the principal axes of the deformation tensor (*16*). Thus, we see from Equation (*4*) that the total rate of change of momentum is resolved into three parts: (a) an instantaneous local rate of increase, $\rho \partial v^j/\partial t$; (b) a rate of change of translational momentum, $\frac{1}{2}\rho g^{jk}(v^2)_{,k}$, and (c) a rate of change of angular momentum of the flow, $\rho g^{jk}\epsilon_{kpq}w^pv^q$.

Since w^j represents the angular velocity of a fluid element, ρw^j is its angular momentum, and ρw^jv^k is the rate at which this angular momentum crosses a unit area with normal direction, n_k. For the case of rectilinear flow, therefore, the operation of vector multiplication defines a vector whose magnitude is proportional to the magnitude of ρw^jv^k and whose direction is normal to the plane defined by w^j and v^k.

Turning our attention now to the resultant forces acting on the element, we see that the only forces which are uniquely related to the motion of the fluid are the surface traction forces which reduce to the hydrostatic pressure, p, when the fluid is quiescent. The surface force term can be written as

$$\int_S T^{jk}n_k dS = \int_V \{-g^{jk}p_{,k} + P^{jk}{}_{,k}\}\, dV \qquad (5)$$

where $-p$ is the mean normal stress field and P^{jk} is the so-called "viscous" stress tensor (*16*) which vanishes identically when $v^j = 0$. The negative of $P^{jk}{}_{,k}$ (often given the symbol $\tau^{jk}{}_{,k}$) represents the rate of loss (*17*) of momentum per unit volume from the element due to the action of viscous tractions. Thus, $P^{jk}{}_{,k}$ is the rate of gain of momentum per unit volume due to the same mechanism.

Turbulent eddy eruptions appear to be essentially nonlinear propagations of unstable rotational motions. Therefore, we surmise that the key mechanism in the development of a flow instability leading to transition to turbulent motion is one of rotational momentum balance. That is, when the distribution of momentum between rotational and frictional modes reaches a certain critical ratio, a rotational instability develops. Since vortex filaments tend to remain intact, this becomes analogous to the flexural whipping of a slender rotating shaft at certain critical speeds, and a kink develops in the vortex filament in the form of a large horse shoe vortex. The fundamental nonlinearity of the system greatly amplifies this unstable motion resulting in the eruption of a high energy turb.

We can express the above concept in terms of momentum fluxes. If the ratio of the magnitude of the rate of change per unit volume of the angular momentum of the deforming fluid to the magnitude of the rate of loss of momentum per unit volume from the fluid through its bounding surfaces exceeds a certain critical value, the deformation becomes unstable in the rotational mode. When this occurs, the nonlinear amplification of certain small rotational disturbances of proper frequency will result in the violent disruption of the system with the attendant eruption of a large, energetic turbulent eddy. This concept can be formulated quantitatively as follows:

$$\frac{\rho[g^{jk}\epsilon_{jpq}w^pv^q\epsilon_{krs}w^rv^s]^{1/2}}{[g_{jk}\tau^{jp}{}_{,p}\tau^{kq}{}_{,q}]^{1/2}} = K \qquad (6)$$

where K is a scalar parameter which depends upon the position vector x^j through the velocity field dependence. When K reaches a certain magnitude κ at some position $\bar{x}^j$, instability develops. Since this theory is phenomenological in origin, the numerical value of κ can only be determined by comparison with experimental data. This was done (*8*) in the earlier treatment where it was discovered that $\kappa = 404$.

DISCUSSION

It is not our purpose in this treatment to compare the theory with experiment, as this has already been done rather extensively (*8* to *13*, *15*). Rather, here we are interested in the general nature of this stability parameter and some of the consequences of the present formulation.

From Equation (*6*), it is clear that K is a variable scalar quantity which is positive definite. For a fixed boundary single-valued velocity field having no inflection points $\tau^{jk}{}_{,k} \neq 0$. Since $v_j(c^k) = 0$, where c^k represents the boundary position vector, it follows that $K(c^k) = 0$ likewise.‡ Similarly, if s^k represents the position vector of the *symmetry* positions in the flow, (where all velocity gradients vanish), then $w^j(s^k) = 0$ and, therefore, $K(s^k) = 0$ also. Consequently, K possesses one or more relative maxima, $\overline{K}$ at locations which are the zeros of $K_{,j} = 0$. When any $\overline{K} \geqq \kappa$, the field is rotationally unstable at these locations. This last result is quite at variance with the intuitive reasoning of some investigators (*18*) who reasoned the point of maximum velocity in a pipe flow field should be the least stable. It is, however, quite in agreement (*8*) with careful observations (*19*, *20*).

Certain special classes of motion are frequently encountered in practice. We shall consider two of them here. The first of these are fixed boundary motions which possess multiple valued velocity fields, an example of which is the field assaociated with thermal convection in vertical pipes (*15*). For such flows $\tau^{jk}{}_{,k}$ possesses one or more zeros within the flow region. From Equation (*6*), it is clear that in the vicinity of zeros of $\tau^{jk}{}_{,k}$, K becomes very large, indicating instability. Such flows have been shown experimentally (*15*) to be unstable at very low bulk Reynolds numbers.

A large class of fixed boundary flow fields of great practical importance consists of steady rectilinear motions. Such fields occur in straight channels of arbitrary but axially invariant cross section. For this special case, it can be shown that $\rho g^{jk}\epsilon_{kpq}w^pv^q = -\frac{1}{2}\rho g^{jk}(v^2)_{,k}$ and $\partial v^j/\partial t = 0$. Consequently, it follows that $\tau^{jk}{}_{,k} = \rho f^j - g^{jk}p_{,k}$ from the equation of motion. Therefore Equation (*6*) assumes the special form

$$K_{\text{rect}} = \tfrac{1}{2}\rho \frac{[g^{kp}v^2{}_{,k}v^2{}_{,p}]^{1/2}}{[\rho^2 g_{jk}f^jf^k - 2\rho f^k p_{,k} + g^{km}p_{,k}p_{,m}]^{1/2}} \qquad (7)$$

The consequence of this result is that for steady rectilinear flows the stability of the flow can equally well be viewed in terms of a balancing of the rate of change of translational momentum (or the gradient of the translational kinetic energy) against the mechanical pressure and external body forces. In fact, this viewpoint is indistinguishable from the angular momentum instability theory. This kinetic energy viewpoint was recognized some time ago for pipe flow of Newtonian (*19*) and more recently non-Newtonian (*21*) fluids. However, it is a special case applicable only to steady single-valued rectilinear flows as has clearly been shown experimentally (*15*). Earlier expressions (*19*, *21*) of this special case were not formulated correctly for other than the particular system being studied by the particular author. Equation (*7*) represents the proper generalization.

At this point, it is instructive to note that the special form of K given by Equation (*7*) can always be written as $N_{Re}F(x^1, x^2)$, where N_{Re} is an arbitrarily defined Reynolds number and $F(x^1, x^2)$ is a geometrical function arising from the numerator for $\mathbf{v} = v^3(x^1, x^2)\mathbf{e}^3$ (the neces-

‡ This, of course, is true only for that class of flows where the fluid adheres to the fixed boundaries. Different reasoning would apply to cases where boundary slip occurs.

sary result of the assumption of rectilinearity).

For this class of flows we have used the fact that $\delta m^j/\delta t = 0$; that is, the flow is nonaccelerative. Consequently, the physical interpretation commonly ascribed (*22, 23*) to the Reynolds number (and by association to K_{rect} from Equation (7) as being (*8*) a ratio of the magnitude of certain inertial forces to the magnitude of the viscous forces acting on a fluid element is clearly meaningless, since for this class of flows the inertial or acceleration force vanishes. However, even though the flow is nonaccelerative, it nonetheless possesses angular momentum, the flux of which is a variable function of position. Therefore, the interpretation of the stability parameter in terms of the angular momentum and frictional drag momentum fluxes which led to Equation (6) is the more fundamentally sound.

From the above formulation and the results of many comparisons with experiment, we see that Equation (6) represents a stability criterion for laminar velocity fields which is quite general in scope. Although the theory is phenomenological in nature, necessitating an appeal to experimental data to fix the critical magnitude of K, it is believed that the basic concept is fundamentally sound. The theory has the great advantage, from a practical standpoint, of being predictive rather than only explanatory. Hopefully this advantage outweighs the lack of complete theoretical rigor inherent in all phenomenological analyses.

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NOTATION

(The notation used in this paper is standard indicial tensorial notation).

f = external body force
g = metric tensor
K = stability parameter, Equation (6)
K_{rect} = special case stability parameter, Equation (7)
m = momentum
n = unit normal vector
p = mean normal stress
P = viscous stress tensor
R = resultant force
S = surface element
T = total stress tensor
v = velocity
V = volume
w = vorticity
x = position vector

Greek Letters

$\partial/\partial t$ = partial time differential operator
$\delta/\delta t$ = intrinsic derivative operator
ϵ = absolute permutation tensor
κ = critical value of K
ρ = fluid density
τ = $-P$

Superscripts

i, j, k = contravarient indices

Subscripts

i, j, k = covarient indices
$,k$ = covarient differentiation operator

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APPENDIX

In the preceding discussion considerable emphasis was placed on the boundaries of the flow field being fixed. The significance of such emphasis becomes more apparent when one considers the problem of flow stability in a Couette system. For planar Couette flow $\tau^{jk}{}_{,k} \equiv 0$ and for the cylindrical Couette viscometer with a narrow gap $\tau^{jk}{}_{,k} = 0$ as well.

In both of these cases the numerator of Equation (6) is nonzero and hence instability is indicated at all velocities. It is well known, however, in the case of the cylindrical Couette device that below a certain critical rotational Reynolds number (*5*) stable flow occurs, and that at the critical rotational Reynolds number a transition occurs to a stable secondary cellular vortex motion.

This behavior is associated with the centripetal acceleration field which is superimposed upon the motion. Therefore, in this case it seems likely that the third mechanism mentioned above is operative, making possible the stable motion of an otherwise unstable flow field.

In the planar Couette case, however, no such superimposed acceleration field is operative. Because of the experimental difficulty of achieving a truly planar Couette flow field, the stability of such flows has not been investigated experimentally.

In the case of the thermal convection instability studied by Scheele and Greene (*15*) the condition $\tau^{jk}{}_{,k} = 0$ correctly predicted transition at very low Reynolds number. One significant physical difference exists between the Couette flows and the other types of flows studied. In the non-Couette systems, even the ones for which $\tau^{jk}{}_{,k} = 0$, the boundaries of the flow field are stationary and the flow, and hence the fluid momentum transfer process is driven by a pressure field or an external body force field. In the Couette systems, on the other hand, the flow is driven by the frictional drag of a moving boundary and no external pressure or body force fields feed momentum into the fluid. As a result of this different momentum supply mechanism, the energy transfer mechanisms differ, particularly with respect to the transfer of baseflow energy to fluctuations. It is significant to observe in the Couette viscometer case that although a transition does occur at a certain critical angular speed, the resultant flow is not turbulent but rather a stabilized secondary cellular motion. Therefore, although Equation (6) clearly indicates instability for the cylindrical Couette case for all rotational speeds, the different energy transfer mechanism caused by the moving boundary apparently stabilizes the flow into at least two nonturbulent modes, the distinction between which requires a different type of analysis (5) to make.

As a result of the above observations it appears that the present theory accurately predicts profile instability and transition to turbulent motion in stationary boundary flows but predicts only velocity profile instability in moving boundary flows. Since the present theory deals only with profile stability, no quantitative predictions can be made concerning the energy transfer mechanism and other theories (5) must be relied upon in moving boundary cases to predict transitions between the stabilized nonturbulent motions.

Graphical Calculation of Multiple Steady States and Effectiveness Factors for Porous Catalysts

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Simple graphical methods are given for predicting the effectiveness factors of single reactions in particles of various shapes. A collocation procedure is used for small particles, and the known asymptotic solution is used for large particles. Multiple steady states and variable fluid properties can be handled directly. Examples are given for several nonlinear reaction rate laws.

The prediction of catalyst effectiveness factors for nonlinear situations is a continuing problem in process design and research. Such calculations can be done routinely by conventional finite-difference techniques, but this approach gives little insight, and permits dangerous misinterpretations if multiple solutions are present. Similar criticisms apply to the use of generalized coordinates to extrapolate effectiveness factor plots from one reaction rate law to another.

Simple calculation methods are presented here for two ranges of particle size. The solution for large particles was derived by Stewart (*12*) for isothermal systems, and extended by Peterson (*10, 11*) to nonisothermal systems. The solution for small particles is obtained from a collocation principle given by Villadsen and Stewart (*13*); it gives detailed information about the number and location of the steady states. From these two asymptotic solutions, useful predictions can be made for all particle sizes.

STATEMENT OF THE PROBLEM

Consider a single chemical reaction at steady state in a porous particle, with n fluid species present. The mass and energy conservation equations are taken as:

$$(\nabla \cdot \mathfrak{D}_i \nabla c_i) = -R_i \quad i = 1, \dots n \tag{1}$$

$$(\nabla \cdot k \nabla T) = -\sum_{i=1}^{n} (\nabla \cdot \overline{H}_i \mathfrak{D}_i \nabla c_i) \tag{2}$$

The production rates of the species per unit particle volume are expressed as

$$R_i = \nu_i R_A (c_1, \dots c_n, T) \quad i = 1, \dots n \tag{3}$$

in which A is a species chosen as a basis for calculations. This equation also defines ν_i, the stoichiometric coefficient for species i relative to A. The conditions on the outer boundary of the particle are considered to be constant:

$$\text{on } S_p, \begin{cases} c_i = c_{i0} \quad i = 1, \dots n & (4) \\ T = T_0 & (5) \end{cases}$$

Expressions for the profiles of temperature and concentration, and the effectiveness factor

$$\eta = \frac{1}{R_{A0} V_p} \iiint R_A \, dV_p, \tag{6}$$

are desired. Here R_{A0} is the production rate of species A at the outer surface conditions, and V_p is the volume of the particle. The quantities $\mathfrak{D}_i$, k, $\overline{H}_i$ and R_A are considered to depend only on $c_1, \dots c_n$ and T; that is, the particle is considered to be homogeneous and isotropic.

The quantities $\mathfrak{D}_i$ and k are effective transport properties, defined by the abbreviated flux equations $\mathbf{N}_i = -\mathfrak{D}_i \nabla c_i$ and $\mathbf{q}^{(c)} + \mathbf{q}^{(x)} = -k \nabla T$. All current theories of transport in porous media can be thus abbreviated, when use is made of the stoichiometric constraints $\mathbf{N}_i = \nu_i \mathbf{N}_A$ and $\mathbf{q}^{(c)} + \mathbf{q}^{(x)} = -\sum_{i=1}^{n} \overline{H}_i \nu_i \mathbf{N}_A$ that hold within the particle. These constraints are consistent with the steady state conservation equations for a homogeneous, isotropic particle under boundary conditions (4) and (5).